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ON THE STRONG LIMITING BEHAVIOR OF LOCAL FUNCTIONALS OF EMPIRICAL PROCESSES BASED UPON CENSORED DATA

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We prove functional laws of the iterated logarithm for empirical processes based upon censored data in the neighborhood of a fixed point. We apply these results to obtain strong laws for estimators of local functionals of the lifetime distribution. In particular, we describe the pointwise strong limiting behavior of the kernel density estimator based upon the Kaplan–Meier product-limit estimator.

1. Introduction. In this paper, we are concerned with nonparametric estimators of the *lifetime density function* and related functionals based on *censored data*. Let the *lifetimes* $\{X_i,\ i \geq 1\}$ and the *censoring times* $\{Y_i,\ i \geq 1\}$ be independent sequences of independent and identically distributed nonnegative random variables. Set $X = X_1$, $Y = Y_1$, $F(x) = P(X \leq x)$ and $G(x) = P(Y \leq x)$. In the literature, the problem of estimating F from censored samples has received much attention [see, e.g., Kalbfleisch and Prentice (1980), Földes, Rejtő and Winter (1981), Gu and Lai (1990) and the references therein]. In the *random censorship from the right model*, one observes the pairs $(Z_i,\ \delta_i)$ for $i = 1, \dots, n$, where $Z_i = \min(X_i,\ Y_i)$ and $\delta_i = \mathbb{1}_{\{X_i \leq Y_i\}}$ with $\mathbb{1}_E$ denoting the indicator function of E. The nonparametric maximum likelihood estimator of $F(z)$ based on this data set is the *product-limit estimator*, introduced by Kaplan and Meier (1958) and defined by

$$F_n(z) = 1 - \prod_{i:\, Z_i \leq z,\, 1 \leq i \leq n} \left(\frac{N_n(Z_i) - 1}{N_n(Z_i)} \right)^{\delta_i}, \tag{1.1}$$

where $N_n(x) = \sum_{i=1}^n \mathbb{1}_{\{Z_i \geq x\}}$, and with the conventions that $\prod_{\varnothing} = 1$ and $0^0 = 1$. In many applications, one needs to estimate *local functionals* of F, typical examples of which are the *density* $f(z)$ of F, and the *failure rate* (or *hazard function*) $f(z)/(1 - F(z))$, assuming that they exist. We refer to Földes, Rejtő and Winter (1981), Schäfer (1986), Yandell (1983), Padgett and McNichols (1984), Liu and Van Ryzin (1985) and Lo, Mack and Wang (1989) for examples of nonparametric estimators of local functionals of this kind. In the sequel, we will consider in more detail the *kernel estimator* f_n of f which, besides being of interest in itself, will illustrate our forthcoming more general

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results. Let $\{h_n, n \geq 1\}$ be positive constants, assumed to satisfy conditions (H1) and (H2):

(H1) $h_n \downarrow 0$, $nh_n \uparrow \infty$;
(H2) $nh_n/\log_2 n \to \infty$, where $\log_2 u := \log(\log \max(u, 3))$.

Let K be a function (or *kernel*) satisfying the following assumptions:

(K1) $K(\cdot)$ is of bounded variation on $(-\infty, \infty)$;
(K2) for some $0 < M < \infty$, $K(u) = 0$ for all $|u| \geq M/2$;
(K3) $\int_{-\infty}^{\infty} K(u)\, du = 1$.

The kernel estimator of $f(z)$ [Watson and Leadbetter (1964a, b), Tanner and Wong (1983)] is

$$f_n(z) = \int_{-\infty}^{\infty} h_n^{-1} K\left(\frac{t-z}{h_n}\right) dF_n(t). \tag{1.2}$$

In this paper, we will obtain the almost-sure rate of consistency of $f_n(z)$ for a fixed z. This will be shown to be a consequence of *functional laws of the iterated logarithm* describing the local oscillations of F_n, which constitute our main results. The following notation and assumptions will be needed to state the corresponding theorems. For any function $R(x)$, we set

$$R_-(x) = \lim_{\varepsilon \downarrow 0} R(x - \varepsilon) \quad \text{and} \quad R_+(x) = \lim_{\varepsilon \downarrow 0} R(x + \varepsilon),$$

whenever these limits exist. For any right-continuous distribution function $L(x) = L_+(x)$, we set $T_L = \sup\{t\colon L(t) < 1\}$, $L_-(\infty) = \lim_{x \to \infty} L(x)$ and $L_-(x) = \lim_{\varepsilon \downarrow 0} L(x - \varepsilon)$. We assume that $F_-(\infty) = 1$, but allow the distribution of Y to be defective with $G_-(\infty) = 1 - P(Y = \infty) \leq 1$. In particular, when $P(Y = \infty) = 1$, we have $G(x) = 0$, for all $x < \infty$, and obtain the *uncensored case*, F_n then being equal to the usual empirical distribution function based upon $X_1, \dots, X_n$.

Setting $\Theta = \min(T_F, T_G)$, we assume that $\Theta > 0$ (when $\Theta = 0$, F_n is degenerate with probability 1). We let

$$f_-(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(F(x - \varepsilon) - F_-(x))$$

[resp., $f_+(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(F(x + \varepsilon) - F(x))$] and

$$g_-(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(G(x - \varepsilon) - G_-(x))$$

[resp., $g_+(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(G(x + \varepsilon) - G(x))$] denote the values of the left derivatives of F_- and G_- (resp., right derivatives of F and G) at $x \in (0, \Theta)$, whenever these quantities exist. Note for further use that, with the definitions above, the existence of $f_-(x)$ and $f_+(x)$ [resp., $g_-(x)$ and $g_+(x)$] does not imply the continuity of F (resp., G) at x. Whenever F (resp., G) is continuous at x, $f_-(x)$ and $f_+(x)$ [resp., $g_-(x)$ and $g_+(x)$] become the usual left and right derivatives of F (resp., G) at x. Set

$$\mathbb{E} f_n(z) = \int_{-\infty}^{\infty} h_n^{-1} K\left(\frac{t-z}{h_n}\right) dF(t). \tag{1.3}$$

With the exception of the uncensored case where $\mathbb{E} f_n(z)$ coincides with the usual expectation $Ef_n(z)$ of $f_n(z)$, we have in general $\mathbb{E} f_n(z) \neq Ef_n(z)$ (even though both expressions may be very close to each other). The limiting behavior of $f_n(z)$ is described in the following theorem.

THEOREM 1.1. *Let $z \in (0, \Theta)$ be fixed. Assume that F is continuous in a neighborhood of z and that the left derivative $f_-(x)$ and right derivative $f_+(x)$ of F at x exist at $x = z$. Then, under* (H1), (H2) *and* (K1)–(K3), *we have*

$$\begin{aligned} \limsup_{n\to\infty} \pm \left\{\frac{nh_n}{2\log_2 n}\right\}^{1/2} (f_n(z) - \mathbb{E} f_n(z)) \\ = \left\{\frac{f_-(z)}{1-G_-(z)} \int_{-\infty}^{0} K^2(t)\,dt \right. \\ \left. + \frac{f_+(z)}{1-G(z)} \int_0^{\infty} K^2(t)\,dt \right\}^{1/2} \quad \text{a.s.} \end{aligned} \tag{1.4}$$

REMARK 1.1. If we assume that the derivative $f(x) = F'(x)$ of F at $x = z \in (0, \Theta)$ exists and that G is continuous at z, then (1.4) may be rewritten as the simpler expression

$$\begin{aligned} \limsup_{n\to\infty} \pm \left\{\frac{nh_n}{2\log_2 n}\right\}^{1/2} (f_n(z) - \mathbb{E} f_n(z)) \\ = \left\{\frac{f(z)}{1-G(z)} \int_{-\infty}^{\infty} K^2(t)\,dt \right\}^{1/2} \quad \text{a.s.} \end{aligned} \tag{1.5}$$

REMARK 1.2. Let F be possibly discontinuous at x but such that the left derivative $f_-(x)$ of F_- and right derivative $f_+(x)$ of F at $x = z \in (0, \Theta)$ exist. Then, under (H1), (K1) and (K2),

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E} f_n(z) = \tfrac{1}{2}(f_+(z) + f_-(z)) \\ + \tfrac{1}{2}(f_+(z) - f_-(z)) \int_0^{\infty} \{K(t) - K(-t)\}\,dt, \end{aligned} \tag{1.6}$$

which, under the additional assumption (K3), reduces to $f(z)$ when $f_-(z) = f_+(z) = f(z)$. In the latter case, we infer from (1.5)–(1.6) that conditions (H1) and (H2) are sufficient for strong consistency of $f_n(z)$ to $f(z)$.

The proof of Theorem 1.1 is postponed until Section 3. A version of this theorem was proved by Lo, Mack and Wang (1989), who established (1.5) under the additional assumptions that (a) G is continuous, (b) K is symmetric and continuous and (c) $nh_n(\log_2 n)/(\log n)^4 \to \infty$, the latter condition (c) being more restrictive than (H2). In the uncensored case, their results essentially coincide with those of Hall (1981), which are therefore also improved by Theorem 1.1. A detailed study of the uncensored case is to be found in Deheuvels and Mason (1994). It follows from their discussion and from earlier results of Deheuvels (1974) that condition (H2) is sharp in the sense that the conclusion of Theorem 1.1 becomes false in general when (H2) is replaced by $nh_n/\log_2 n \to c \in [0, \infty)$. The growth and regularity conditions in (H1) may be relaxed in part by making use of an auxiliary sequence (see Remark 3.1 in the sequel).

Lo, Mack and Wang (1989) follow Lo and Singh (1986) and base their arguments on the strong invariance principles of Burke, Csörgő and Horváth (1981, 1988). On the other hand, our approach is novel in the sense that it does not make use of invariance principles. Theorem 1.1 and other results of the kind will be instead shown to be consequences of a *local functional law of the iterated logarithm* stated in Theorem 1.2. Introduce the *Kaplan–Meier empirical process* by setting, for $n \geq 1$ and $-\infty < x < \infty$,

$$a_n(x) = n^{1/2}(F_n(x) - F(x)). \tag{1.7}$$

Fix $z \in (0, \Theta)$, and consider the sequence of random functions defined for $n \geq 1$ by

$$\xi_n(u) = b_n^{-1}(a_n(z + h_n u) - a_n(z)) \quad \text{for } -M \leq u \leq M, \tag{1.8}$$

where $M > 0$ is a specified constant and $b_n = (2h_n \log_2 n)^{1/2}$. If $\mathscr{I}$ is an arbitrary set, we denote by $\mathbb{B}(\mathscr{I})$ the space of all bounded real-valued functions defined on $\mathscr{I}$, endowed with the topology of uniform convergence on $\mathscr{I}$. Our main result is as follows.

THEOREM 1.2. *Let $z \in (0, \Theta)$ be fixed. Assume that F is continuous in a neighborhood of z and that the left derivative $f_-(x)$ and right derivative $f_+(x)$ of F at $x = z$ exist. Then, under* (H1) *and* (H2), *the sequence $\{\xi_n: n \geq 1\}$ is almost surely relatively compact in $\mathbb{B}([-M, M])$ with limit set equal to the set of all functions $h \in \mathbb{B}([-M, M])$ of the form*

$$h(u) = \int_0^u \Psi(s)\,ds \quad \text{for } -M \leq u \leq M, \tag{1.9}$$

with

$$\left(\frac{1 - G_-(z)}{f_-(z)}\right)\int_{-M}^0 \Psi^2(s)\,ds + \left(\frac{1 - G(z)}{f_+(z)}\right)\int_0^M \Psi^2(s)\,ds \leq 1.$$

REMARK 1.3. (i) In (1.9), we use the conventions that

$$\int_{-M}^{0} \Psi^2(s)\,ds = 0 \quad \text{and} \quad \left(\frac{1-G_-(z)}{f_-(z)}\right)\int_{-M}^{0} \Psi^2(s)\,ds = 0 \quad \text{when } f_-(z) = 0,$$

and

$$\int_{0}^{M} \Psi^2(s)\,ds = 0 \quad \text{and} \quad \left(\frac{1-G(z)}{f_+(z)}\right)\int_{0}^{M} \Psi^2(s)\,ds = 0 \quad \text{when } f_+(z) = 0.$$

(ii) If we assume that the derivative $f(x) = F'(x) = f_-(x) = f_+(x)$ of F at $x = z \in (0, \Theta)$ exists and that G is continuous at z, then (1.9) may be rewritten as the simpler expression

$$h(u) = \int_0^u \Psi(s)\,ds \quad \text{for } -M \le u \le M, \qquad \text{with } \int_{-M}^{M} \Psi^2(s)\,ds \le \frac{f(z)}{1-G(z)}. \tag{1.10}$$

We will present the proof of Theorem 1.2 in Section 2, together with other local functional laws of the kind based upon $\{(Z_i, \delta_i), 1 \le i \le n\}$. In particular, we will consider in this section the case where F is possibly discontinuous. A rough outline of our arguments is as follows. Our theorems will be obtained by combining two main ingredients. The first, due to Einmahl and Koning (1992), expresses the main empirical functionals of the censored data in terms of an auxiliary uniform empirical process. The second, due to Deheuvels and Mason (1994), is a local functional law of the iterated logarithm which, when applied to the previous uniform empirical process, will yield the results we seek.

In Section 3, we will show how Theorem 1.2 and the results of Section 2 may be applied to describe the pointwise limiting behavior of a large class of nonparametric local estimators.

2. Functional laws for censored processes. The following notation and assumptions will be in force, in addition to those previously given in Section 1. We first assume the distribution functions $F(x) = P(X \le x)$ of $X = X_1$ and $G(x) = P(Y \le x)$ of $Y = Y_1$ to be possibly discontinuous. Let $\delta = \delta_1$ and $Z = Z_1 = \min(X, Y)$. The right-continuous distribution function of Z, denoted by

$$\begin{aligned} H(x) &= P(Z \le x) = 1 - (1 - F(x))(1 - G(x)) \\ &= H^{(1)}(x) + H^{(0)}(x), \end{aligned} \tag{2.1}$$

is decomposed into the sum of

$$H^{(1)}(x) = P(Z \le x \text{ and } \delta = 1) = \int_0^x (1 - G_-(t))\,dF(t), \tag{2.2}$$

and

$$H^{(0)}(x) = P(Z \le x \text{ and } \delta = 0) = \int_0^x (1 - F(t))\, dG(t). \tag{2.3}$$

Set

$$p = P(\delta = 1) = \int_0^\infty (1 - G_-(t))\, dF(t) = H_-^{(1)}(\infty) = 1 - H_-^{(0)}(\infty). \tag{2.4}$$

Our assumptions exclude $p = 0$, whereas $p = 1$ is possible and corresponds to *uncensored data*. It will become obvious later on that, in the latter case, the results of this section are direct consequences of similar theorems for the uniform empirical process, due to Deheuvels and Mason (1994). Therefore, we will assume from now on without loss of generality that $0 < p < 1$. Let

$$Q^{(1)}(s) = \inf\{x: H^{(1)}(x) \ge s\} \quad \text{for } 0 < s < p, \tag{2.5}$$

and

$$Q^{(0)}(s) = \inf\{x: H^{(0)}(x) \ge s\} \quad \text{for } 0 < s < 1 - p. \tag{2.6}$$

Note for further use that definitions (2.2)–(2.3) and (2.5)–(2.6) entail that

$$\begin{aligned} Q^{(1)}(s) \le x \Leftrightarrow s \le H^{(1)}(x) \quad &\text{for } 0 < s < p, \\ Q^{(0)}(s) \le x \Leftrightarrow s \le H^{(0)}(x) \quad &\text{for } 0 < s < 1 - p. \end{aligned} \tag{2.7}$$

The empirical counterparts of H and $H^{(j)}$, $j = 0, 1$, are obtained by setting

$$H_n(x) = n^{-1} \sum_{i=1}^n \mathbb{1}_{\{Z_i \le x\}} = 1 - n^{-1} N_{n-}(x) = H_n^{(1)}(x) + H_n^{(0)}(x), \tag{2.8}$$

where $N_{n-}(x) = \lim_{\varepsilon \downarrow 0} N_n(x - \varepsilon)$ and N_n is as in (1.1),

$$H_n^{(1)}(z) = n^{-1} \sum_{i=1}^n \delta_i \mathbb{1}_{\{Z_i \le z\}} \quad \text{and} \quad H_n^{(0)}(z) = n^{-1} \sum_{i=1}^n (1 - \delta_i) \mathbb{1}_{\{Z_i \le z\}}. \tag{2.9}$$

The following lemma, inspired by an observation of Einmahl and Koning (1992), will play an instrumental role in our proofs. Let $Z = Z_1$ and $\delta = \delta_1$.

LEMMA 2.1. *On a suitably enlarged probability space, it is possible to define a* uniform(0, 1) *random variable U such that, with probability* 1,

$$\begin{aligned} &\text{(i)}\ \ \delta = \mathbb{1}_{\{0 < U < p\}} = 1 - \mathbb{1}_{\{p < U < 1\}}; \\ &\text{(ii)}\ Z = Q^{(1)}(U) \qquad \textit{when } 0 < U < p; \\ &\qquad Z = Q^{(2)}(U - p) \quad \textit{when } p < U < 1. \end{aligned} \tag{2.10}$$

PROOF. When both distribution functions $H^{(1)}$ and $H^{(0)}$ are continuous, we may set, as in Einmahl and Koning (1992), $U = \delta H^{(1)}(Z) + (1 - \delta)(p + H^0(Z))$, which is readily checked to follow a uniform distribution on (0, 1) and to satisfy (2.10) with probability 1. When $H^{(1)}$ and $H^{(0)}$ are arbitrary, the following argument is needed. First, we observe that, whenever V is a random variable with distribution function $D(v) = P(V \le v)$ and quantile function $D^{\mathrm{inv}}(s) = \inf\{v\colon\ D(v) \ge s\}$ for $0 < s < 1$, we may define V on a probability space which carries a random variable W uniformly distributed on (0, 1) such that $V = D^{\mathrm{inv}}(W)$ with probability 1.

Next, we infer from (2.2) and (2.4) that the conditional distribution function of Z given that $\delta = 1$ is equal to $p^{-1}H^{(1)}(x)$, and from (2.5) that the corresponding quantile function is equal to $Q^{(1)}(ps)$ for $0 < s < 1$. Therefore, an application of the above observation shows that, conditionally on $\delta = 1$, there exists a uniform(0, 1) random variable W_1 such that $Z = Q^{(1)}(pW_1)$ with probability 1. Likewise, conditionally on $\delta = 0$, we may define a uniform(0, 1) random variable W_2 such that $Z = Q^{(2)}((1-p)W_2)$ with probability 1. We conclude by the observation that the random variable $U = \delta p W_1 + (1-\delta)(p + (1-p)W_2)$ is uniformly distributed on (0, 1) and satisfies (2.10) with probability 1. □

By Lemma 2.1 and (2.10), we can and do assume that the original sequence is defined on a probability space which carries a sequence $\{U_n,\ n \ge 1\}$ of independent uniform(0, 1) random variables such that, almost surely for each $i = 1, 2, \dots,$

$$\text{(2.11)}\qquad \begin{aligned} &\text{(i)}\ \ \delta_i = \mathbb{1}_{\{0 < U_i < p\}} = 1 - \mathbb{1}_{\{p < U_i < 1\}};\\ &\text{(ii)}\ Z_i = Q^{(1)}(U_i) \qquad \text{when}\ \ 0 < U_i < p;\\ &\qquad Z_i = Q^{(2)}(U_i - p) \quad \text{when}\ p < U_i < 1. \end{aligned}$$

Consider now the *empirical distribution function*

$$\text{(2.12)}\qquad \mathbb{U}_n(s) = n^{-1}\sum_{i=1}^{n} \mathbb{1}_{\{U_i \le s\}}$$

and the *empirical process* based upon $U_1, \dots, U_n$, denoted by

$$\text{(2.13)}\qquad \alpha_n(s) = n^{1/2}(\mathbb{U}_n(s) - s).$$

In view of (2.7), (2.9) and (2.11) and of definitions (2.12) and (2.13), we see that, almost surely,

$$\text{(2.14)}\qquad H_n^{(1)}(x) = \mathbb{U}_n\big(H^{(1)}(x)\big) \quad \text{for } 0 < H^{(1)}(x) < p,$$

and

$$\text{(2.15)}\qquad H_n^{(0)}(x) = \mathbb{U}_n(H^0(x) + p) - \mathbb{U}_n(p) \quad \text{for } 0 < H^{(0)}(x) < 1 - p.$$

Now let $s_1 \in (0,\ p)$, $s_0 \in (p,\ 1)$ and $M_1 > 0$ be fixed, and consider the sequences of random functions defined for $j = 0, 1$ and $n \ge 1$ by

$$\text{(2.16)}\qquad f_n^{(j)}(u) = b_n^{-1}\big(\alpha_n(s_j + h_n u) - \alpha_n(s_j)\big) \quad \text{for } -M_1 \le u \le M_1,$$

where $b_n = (2h_n \log_2 n)^{1/2}$. Further, set

$$(2.17)\qquad \begin{aligned} w_n^{(1)} &= (2\log_2 n)^{-1/2}\alpha_n(s_1),\\ w_n^{(0)} &= (2\log_2 n)^{-1/2}(\alpha_n(s_0) - \alpha_n(p)). \end{aligned}$$

The next lemma is due to Deheuvels and Mason (1994).

LEMMA 2.2. *Under* (H1) *and* (H2), *the sequence* $\{(w_n^{(1)}, w_n^{(0)}, f_n^{(1)}, f_n^{(0)})\}$ *is almost surely relatively compact in* $\mathbb{R}^2 \times \mathbb{B}^2([-M_1, M_1])$ *with limit set equal to the set of all* $(w^{(1)}, w^{(0)}, f^{(1)}, f^{(0)})$ *with*

$$(2.18)\qquad \begin{aligned} w^{(j)} &= \int_0^{s_j} \Psi(s)\, ds,\\ f^{(j)}(u) &= \int_0^u \Psi_j(s)\, ds \quad \text{for } j = 1, 0 \text{ and } -M_1 \le u \le M_1, \end{aligned}$$

where

$$(2.19)\qquad \int_0^1 \Psi(s)\, ds = 0 \quad \text{and} \quad \int_0^1 \Psi^2(s)\, ds + \int_{-M_1}^{M_1} (\Psi_1^2(s) + \Psi_0^2(s))\, ds \le 1.$$

PROOF. The fact that the sequence $\{f_n^{(1)}(v)\colon 0 \le v \le M_1\}$ is almost surely relatively compact in $\mathbb{B}([0, M_1])$ with limit set as in (2.18)–(2.19) was proved for $s_1 = 0$ by Mason (1988). The corresponding result for $(w_n^{(1)}, w_n^{(0)})$ follows from the Finkelstein (1971) functional law of the iterated logarithm. The extension of these partial results as stated in the lemma is a particular case of Theorem 1.2 of Deheuvels and Mason (1994). It is noteworthy that the proof of the latter theorem does not make use of invariance principles. □

In the sequel, we will make use of a modified version of Lemma 2.2 stated in Lemma 2.3. Fix any $M > 0$ and let $\{\gamma_n^{(j)}(u)\colon -M \le u \le M\}$, $j = 0, 1$, be sequences of functions such that

$$(2.20)\qquad \lim_{n\to\infty} \left\{ \sup_{u\in[-M,M]} \left|\gamma_n^{(j)}(u) - \gamma_j u\right| \right\} = 0,$$

where $\gamma_j > 0$, $j = 0, 1$, are constants. Further, for $j = 0, 1$, let

$$(2.21)\qquad g_n^{(j)}(u) = f_n^{(j)}\left(\gamma_n^{(j)}(u)\right) = b_n^{-1}\left(\alpha_n\left(s_j + h_n\gamma_n^{(j)}(u)\right) - \alpha_n(s_j)\right) \quad \text{for } -M \le u \le M.$$

LEMMA 2.3. *Under* (H1) *and* (H2), *the sequence* $(w_n^{(1)}, w_n^{(0)}, g_n^{(1)}, g_n^{(0)})$ *is almost surely relatively compact in* $\mathbb{R}^2 \times \mathbb{B}^2([-M, M])$ *with limit set equal to the set of all* $(w^{(1)}, w^{(0)}, g^{(1)}, g^{(0)})$ *with*

$$(2.22)\qquad w^{(1)} = \int_0^{s_1} \phi(s)\, ds, \qquad w^{(0)} = \int_p^{s_0} \phi(s)\, ds, \qquad g^{(j)}(u) = \int_0^u \phi_j(s)\, ds,$$

for $j = 0, 1$ *and* $-M \le u \le M$, *where*

$$\int_0^1 \phi(s)\,ds = 0 \quad \text{and}$$
$$(2.23)\qquad \int_0^1 \phi^2(s)\,ds + \int_{-M}^{M} \left(\gamma_1^{-1}\phi_1^2(s) + \gamma_0^{-1}\phi_0^2(s)\right) ds \le 1.$$

PROOF. Choose $M_1 > 0$ in Lemma 2.2 so large that $|\gamma_n^{(j)}(u)| \le M_1$ uniformly over $-M \le u \le M$ and $j = 0, 1$, for all large n. By this lemma, for any increasing sequence $\{n_k\}$ of positive integers, there exists almost surely an increasing subsequence $\{n'_k\}$ of $\{n_k\}$ and a $(w^{(1)}, w^{(0)}, f^{(1)}, f^{(0)})$ as in (2.18) and (2.19) such that, as $k \to \infty$ and along $\{n'_k\}$,

$$(2.24)\qquad \sum_{j=0,1} \left\{ |w_n^{(j)} - w^{(j)}| + \sup_{v \in [-M_1, M_1]} |f_n^{(j)}(v) - f^{(j)}(v)| \right\} \to 0,$$

whence, as $k \to \infty$ and along $\{n'_k\}$,

(2.25)
$$\sum_{j=0,1} \left\{ |w_n^{(j)} - w^{(j)}| + \sup_{u \in [-M, M]} \left| f_n^{(j)}\big(\gamma_n^{(j)}(u)\big) - f^{(j)}\big(\gamma_n^{(j)}(u)\big) \right| \right\} \to 0.$$

Next, we observe that the functions $f^{(j)}$, $j = 0, 1$, in (2.18) and (2.19) are uniformly equicontinuous, as follows from the Schwarz inequality and (2.19), which entail that, for any $-M_1 \le v', v'' \le M_1$,

$$(2.26)\qquad \begin{aligned} |f^{(j)}(v') - f^{(j)}(v'')| &= \left| \int_{v'}^{v''} \Psi_j(s)\,ds \right| \\ &\le |v' - v''|^{1/2} \times \left| \int_{v'}^{v''} \Psi_j^2(s)\,ds \right|^{1/2} \\ &\le |v' - v''|^{1/2}. \end{aligned}$$

By combining (2.20)–(2.21) and (2.25)–(2.26), we obtain readily that, along $\{n'_k\}$,

$$(2.27)\qquad \sum_{j=0,1} \left\{ |w_n^{(j)} - w^{(j)}| + \sup_{u \in [-M, M]} |g_n^{(j)}(u) - f^{(j)}(\gamma_i u)| \right\} \to 0.$$

Recalling (2.18), we set $g^{(j)}(u) = f^{(j)}(\gamma_j u)$ and $\phi_j(u) = \gamma_j \Psi_j(\gamma_j u)$ for $-M \le u \le M$ and $j = 0, 1$. It is readily verified from (2.18)–(2.19) that $(w^{(1)}, w^{(0)}, g^{(1)}, g^{(0)})$ satisfies (2.22)–(2.23). This proves that the sequence $\{(w_n^{(1)}, w_n^{(0)}, g_n^{(1)}, g_n^{(0)})\}$ is almost surely relatively compact with limit set included in the set characterized by (2.22)–(2.23). A similar argument, which we omit for the sake of conciseness, proves that, almost surely, the latter two sets are equal. □

In view of (2.5) and (2.6), we fix $z_1 \in (0, \Theta)$ and $z_0 \in (0, \Theta)$ and set

$$(2.28)\qquad s_1 = H^{(1)}(z_1) \in (0, p), \qquad s_0 = p + H^{(0)}(z_0) \in (p, 1).$$

Fix $M > 0$. For $j = 0, 1$, introduce the sequences of random functions of $u \in [-M, M]$

$$k_n^{(j)}(u) = b_n^{-1} n^{1/2}\big(H_n^{(j)}(z_j + h_n u) - H^{(j)}(z_j + h_n u) - H_n^{(j)}(z_j) + H^{(j)}(z_j)\big) \tag{2.29}$$

and the sequences of random variables

$$v_n^{(j)} = (2\log_2 n)^{-1/2} n^{1/2}\big(H_n^{(j)}(z_j) - H^{(j)}(z_j)\big). \tag{2.30}$$

By (H1), there exists an $n_0 \geq 1$ such that, for all $n \geq n_0$, $j = 0, 1$, and $-M \leq u \leq M$,

$$z_j + h_n u \in (0, \Theta). \tag{2.31}$$

It follows from (2.31) that, for all $n \geq n_0$, $(v_n^{(1)}, v_n^{(0)}, k_n^{(1)}, k_n^{(0)}) \in \mathbb{R}^2 \times \mathbb{B}^2([-M, M])$ is properly defined by (2.30)–(2.31). The following theorem describes the almost-sure limiting behavior of this sequence.

THEOREM 2.1. *Assume that $F(x)$ is continuous and that the derivative $g(x) = G'(x)$ of G exists, with $g(x) > 0$ at $x = z_0 \in (0, \Theta)$. Assume further that $F(x)$ and $G(x)$ are continuous and that the derivative $f(x) = F'(x)$ of F exists, with $f(x) > 0$ at $x = z_1 \in (0, \Theta)$. Then, under* (H1) *and* (H2), *the sequence $\{(v_n^{(1)}, v_n^{(0)}, k_n^{(1)}, k_n^{(0)})\colon n \geq 1\}$ is almost surely relatively compact in $\mathbb{R}^2 \times \mathbb{B}^2([-M, M])$ with limit set equal to the set of all $(v^{(1)}, v^{(0)}, k^{(1)}, k^{(0)}) \in \mathbb{R}^2 \times \mathbb{B}^2([-M, M])$ with*

$$v^{(1)} = \int_0^{H^{(1)}(z_1)} \phi(s)\, ds, \qquad v^{(0)} = \int_p^{p + H^{(0)}(z_0)} \phi(s)\, ds, \qquad k^{(j)}(u) = \int_0^u \phi_j(s)\, ds, \tag{2.32}$$

for $j = 0, 1$ and $-M \leq u \leq M$, where

$$\int_0^1 \phi(s)\, ds = 0 \quad \text{and} \quad \int_0^1 \phi^2(s)\, ds + \int_{-M}^{M} \left(\frac{\phi_1^2(s)}{f(z_1)(1 - G(z_1))} + \frac{\phi_0^2(s)}{g(z_0)(1 - F(z_0))} \right) ds \leq 1. \tag{2.33}$$

PROOF. For $j = 0, 1$, let s_j be as in (2.28) and let $f_n^{(j)}$ be as in (2.16). It follows from (2.14), (2.15), (2.16), (2.28) and (2.29) that

$$\begin{aligned} k_n^{(1)}(u) &= b_n^{-1}\big(\alpha_n\big(H^{(1)}(z_1 + h_n u)\big) - \alpha_n\big(H^{(1)}(z_1)\big)\big) \\ &= f_n^{(1)}\big(h_n^{-1}\big(H^{(1)}(z_1 + h_n u) - H^{(1)}(z_1)\big)\big), \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} k_n^{(0)}(u) &= b_n^{-1}\big(\alpha_n\big(p + H^{(0)}(z_0 + h_n u)\big) - \alpha_n\big(p + H^{(0)}(z_0)\big)\big) \\ &= f_n^{(0)}\big(h_n^{-1}\big(H^{(0)}(z_0 + h_n u) - H^{(0)}(z_0)\big)\big). \end{aligned} \tag{2.35}$$

Likewise, recalling (2.17), we see that

$$
\begin{aligned}
v_n^{(1)} &= (2\log_2 n)^{-1/2}\alpha_n(s_1) = w_n^{(1)} \quad \text{and} \\
v_n^{(0)} &= (2\log_2 n)^{-1/2}(\alpha_n(s_0) - \alpha_n(p)) = w_n^{(0)}.
\end{aligned} \tag{2.36}
$$

For $j = 0, 1$ and $-M \le u \le M$, let

$$\gamma_n^{(i)}(u) = h_n^{-1}\big(H^{(j)}(z_i + h_n u) - H^{(j)}(z_i)\big). \tag{2.37}$$

Consider first $H^{(1)}$ as defined in (2.2). Assuming only that the right derivative $f_+(z_1)$ of F at z_1 exists, the right-continuity of $G(z_1) = G_+(z_1)$ at z_1 entails that

$$
\begin{aligned}
\lim_{h\downarrow 0} h^{-1}\Big\{ \sup_{0\le u\le M} \big|H^{(1)}(z_1 + hu) - H^{(1)}(z_1)& \\
-f_+(z_1)(1 - G(z_1))hu\big|\Big\} &= 0.
\end{aligned} \tag{2.38}
$$

Likewise, assuming only that the left derivative $f_-(z_1)$ of F_- at z_1 exists, we have

$$
\begin{aligned}
\lim_{h\downarrow 0} h^{-1}\Big\{ \sup_{0\le u\le M} \big|H^{(1)}(z_1 - hu) - H_-^{(1)}(z_1)& \\
+f_-(z_1)(1 - G_-(z_1))hu\big|\Big\} &= 0.
\end{aligned} \tag{2.39}
$$

We now observe that the assumptions of the theorem imply that (a) $f(z_1) = f_+(z_1) = f_-(z_1)$ exists, (b) $F(z_1) = F_+(z_1) = F_-(z_1)$, (c) $G(z_1) = G_+(z_1) = G_-(z_1)$ and, by (H1), (d) $h_n \to 0$. In view of the definitions (2.2) of $H_n^{(1)}$ and (2.37) of $\gamma_n^{(1)}$, (a)–(c) imply that $H^{(1)}(z_1) = H_-^{(1)}(z_1)$ and, in view of (2.38)–(2.39), that the function $\gamma_n^{(1)}$ in (2.37) satisfies (2.20) for $j = 1$ and

$$\gamma_1 = f(z_1)(1 - G(z_1)). \tag{2.40}$$

By an argument similar to (2.38)–(2.39), with the formal replacements of $H^{(0)}$, F, G, f, g, z_1 and (2.2) by $H^{(1)}$, G, F, g, f, z_0 and (2.3), respectively, we obtain likewise that, under the assumptions of the theorem, the function $\gamma_n^{(0)}$ in (2.37) satisfies (2.20) for $j = 0$ and

$$\gamma_0 = g(z_0)(1 - F(z_0)). \tag{2.41}$$

In view of (2.34)–(2.35), (2.36)–(2.37) and (2.40)–(2.41), the proof of the theorem is now immediate by a direct application of Lemma 2.3 □

REMARK 2.1. (i) It follows from the arguments in the above proof of Theorem 2.1 that the assumptions of (1) existence and positiveness of $g(x)$ at $x = z_0 \in (0, \Theta)$ and (2) continuity of $F(x)$ at $x = z_0 \in (0, \Theta)$ are only needed to ensure the almost-sure relative compactness of the sequence $\{k_n^{(0)}\}$. To obtain only the almost-sure relative compactness of the sequence $\{k_n^{(1)}\}$ in $\mathbb{B}([-M, M])$ with limit set consisting of all functions $k^{(1)} \in \mathbb{B}([-M, M])$ of the form

$$k^{(1)}(u) = \int_0^u \phi_1(s)\,ds \quad \text{with} \int_{-M}^{M} \phi_1^2(s)\,ds \le f(z_1)(1 - G(z_1)), \tag{2.42}$$

it is enough to assume (3) existence of $f(x) \geq 0$ at $x = z_1 \in (0, \Theta)$ and (4) continuity of $G(x)$ at $x = z_1 \in (0, \Theta)$.

(ii) If we only assume (5) continuity of $F(x)$ at $x = z_1 \in (0, \Theta)$, (6) existence of $f_+(x)$ and $f_-(x)$ at $x = z_1 \in (0, \Theta)$ and we allow G to be possibly discontinuous at $z_1 \in (0, \Theta)$, then the sequence $\{k_n^{(1)}\}$ is almost surely relatively compact in $\mathbb{B}([-M, M])$ with limit set consisting of all functions $k^{(1)} \in \mathbb{B}([-M, M])$ of the form

$$
\begin{aligned}
&k^{(1)}(u) = \int_0^u \phi_1(s)\, ds \\
(2.43)\qquad &\text{with } \int_{-M}^0 \frac{\phi_1^2(s)}{f_-(z_1)(1 - G_-(z_1))}\, ds + \int_0^M \frac{\phi_1^2(s)}{f_+(z_1)(1 - G(z_1))}\, ds \leq 1,
\end{aligned}
$$

with the convention that $\phi_1(s) = 0$ *for* $-M \leq s \leq 0$ *(resp.,* $0 \leq s \leq M$*) when* $f_-(z_1) = 0$ *[resp.,* $f_+(z_1) = 0$*].*

(iii) If we only assume (6) existence of $f_+(x)$ and $f_-(x)$ at $x = z_1 \in (0, \Theta)$ and we allow both F and G to be possibly discontinuous at $z_1 \in (0, \Theta)$, we may infer from (2.34) that $k_{n-}^{(1)}(0) - k_n^{(1)}(0) = f_n^{(1)}(h_n^{-1}(H_-^{(1)}(z_1) - H^{(1)}(z_1))) \nrightarrow 0$ a.s. when $H_-^{(1)}(z_1) - H^{(1)}(z_1) = P(Y \geq z_1)P(X = z_1) \neq 0$, so that the conclusion of Theorem 2.1 is not valid in this case. However, if we set $\kappa_n^{(1)}(u) = k_n^{(1)}(u)$ for $u \geq 0$, and $\kappa_n^{(1)}(u) = k_{n-}^{(1)}(u)$ for $u < 0$, it holds that the sequence $\{\kappa_n^{(1)}\}$ is almost surely relatively compact in $\mathbb{B}([-M, M])$ with limit set consisting of all functions $k^{(1)} \in \mathbb{B}([-M, M])$ satisfying (2.43).

In view of (2.2), (2.34), (2.38) and (2.39), the proofs of the variants of Theorem 2.1 stated in Remark 2.1 are readily achieved along the same lines as the just-given proof of this theorem. The fact that the positivity of $f(z_1)$ is not needed for (2.42) follows from a simple modification of our arguments showing that, when $f(z_1) = 0$,

$$(2.44)\qquad \sup_{u \in [-M, M]} |k_n^{(1)}(u)| \to 0 \quad \text{a.s.}$$

A similar argument shows that (2.43) holds when either $f_-(z_1) = 0$ or $f_-(z_1) = 0$. Recalling the definitions (2.1), (2.8) and (2.29)–(2.30), and letting $z_1 = z_0 = z \in (0, \Theta)$, we now introduce the sequence of random functions of $u \in [-M, M]$ defined for $n \geq 1$ by

$$
\begin{aligned}
(2.45)\qquad k_n(u) &= k_n^{(1)}(u) + k_n^{(0)}(u) \\
&= b_n^{-1} n^{1/2}(H_n(z + h_n u) - H(z + h_n u) - H_n(z) + H(z)),
\end{aligned}
$$

and the sequence of random variables

$$(2.46)\qquad v_n = v_n^{(1)} + v_n^{(0)} = (2 \log_2 n)^{-1/2} n^{1/2}(H_n(z) - H(z)).$$

The following corollary will be shown to be an easy consequence of Theorem 2.1.

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1 *with* $z = z_0 = z_1 \in (0, \Theta)$, *the sequence* $\{(v_n, k_n): n \geq 1\}$ *is almost surely relatively compact in* $\mathbb{R} \times \mathbb{B}([-M, M])$ *with limit set equal to the set of all* $(v, k) \in \mathbb{R} \times \mathbb{B}([-M, M])$, *with*

$$(2.47)\quad v = \int_0^{H(z)} \phi(s)\,ds, \qquad k(u) = \int_0^u \Phi(s)\,ds \quad \text{for } -M \leq u \leq M,$$

where

$$(2.48)\quad \begin{gathered} \int_0^1 \phi(s)\,ds = 0 \quad \text{and} \\ \int_0^1 \phi^2(s)\,ds + \int_{-M}^{M} \left(\frac{\Phi^2(s)}{f(z)(1-G(z)) + g(z)(1-F(z))} \right) ds \leq 1. \end{gathered}$$

PROOF. By Theorem 2.1, it is enough to check that, for $z_1 = z_0 = z$, the image set $\mathscr{J}(\mathscr{A})$ by the mapping $\mathscr{J}$: $(v^{(1)}, v^{(0)}, k^{(1)}, k^{(0)}) \to (v = v^{(1)} + v^{(0)}, k = k^{(1)} + k^{(0)})$ of the set $\mathscr{A}$ characterized by (2.32)–(2.33) is equal to the set $\mathscr{B}$ characterized by (2.47)–(2.48). Toward the aim of proving that $\mathscr{J}(\mathscr{A}) = \mathscr{B}$, we observe that the infimum of $x^2/a^2 + y^2/b^2$ given that $x + y = \rho$ is equal to $\rho^2/(a^2 + b^2)$ and is reached for $x = a^2\rho/(a^2 + b^2)$ and $y = b^2\rho/(a^2 + b^2)$. By letting $a^2 = f(z)(1 - G(z))$, $b^2 = g(z)(1 - F(z))$ and $\rho = \Phi(s)$, we see that the choice of $\phi_1(s) = x$ and $\phi_0(s) = y$ for each $s \in [-M, M]$ ensures first that $\Phi(s) = \phi_1(s) + \phi_0(s)$ and second that

$$(2.49)\quad \begin{aligned} &\int_{-M}^{M} \left(\frac{\phi_1^2(s)}{f(z)(1-G(z))} + \frac{\phi_0^2(s)}{g(z)(1-F(z))} \right) ds \\ &\qquad = \int_{-M}^{M} \left(\frac{\Phi^2(s)}{f(z)(1-G(z)) + g(z)(1-F(z))} \right) ds. \end{aligned}$$

This in turn is sufficient to show that each (v, k) belonging to $\mathscr{B}$ is the image by $\mathscr{J}$ of some $(v^{(1)}, v^{(0)}, k^{(1)}, k^{(0)}) \in \mathscr{A}$. The proof that each $(v^{(1)}, v^{(0)}, k^{(1)}, k^{(0)}) \in \mathscr{A}$ is mapped into $\mathscr{B}$ by $\mathscr{J}$ is similar and therefore omitted. □

The following sequence of lemmas is directed toward the proof of Theorem 1.2. For the sake of conciseness and notational simplicity, we will assume from now on and unless otherwise specified that F is continuous in a neighborhood of z_1, that G is continuous at z_1 and that the derivative

$f(x) = F'(x)$ of F exists at $x = z_1 \in (0, \Theta)$. It will become obvious later on that our arguments can be applied with minor modifications to cover the case where only the left and right derivatives $f_-(z_1)$ and $f_+(z_1)$ of F exist (see Remark 2.1 for the statement of the corresponding variants). In view of (2.2), it is noteworthy that the assumptions above imply that $F_-(x) = F(x)$ and $H_-^{(1)}(x) = H^{(1)}(x)$ for all x in a neighborhood of z_1.

We first consider the so-called *basic martingale* [see, e.g., Gu and Lai (1990), (2.1), page 167],

$$M_n(x) = n\left(H_n^{(1)}(x) - \int_0^x (1 - H_{n-}(s))\, d\Lambda(s)\right), \tag{2.50}$$

where $\Lambda(s) = -\log(1 - F(s))$ and, in view of (1.1) and (2.8), $H_{n-}(s) = 1 - n^{-1}N_n(s)$ is the left-continuous version of H_n [this terminology follows from the fact [see, e.g., Aalen (1976)] that $M_n(x)$ is a martingale with respect to the filtration $\mathbb{F}_x = \sigma\{Z_i \mathbb{1}_{\{Z_i \le x\}}, \delta_i \mathbb{1}_{\{Z_i \le x\}}: i = 1, \dots, n\}$]. In view of (2.1) and (2.2), we may write

$$H^{(1)}(x) = \int_0^x (1 - F(s))(1 - G_-(s))\, d\Lambda(s) = \int_0^x (1 - H_-(s))\, d\Lambda(s). \tag{2.51}$$

It follows from (2.50) and (2.51) that

$$\begin{aligned} n^{-1/2}M_n(x) &= n^{1/2}\big(H_n^{(1)}(x) - H^{(1)}(x)\big) \\ &\quad + n^{1/2}\int_0^x (H_{n-}(s) - H_-(s))\, d\Lambda(s). \end{aligned} \tag{2.52}$$

Fix $z_1 \in (0, \Theta)$ and consider the increment functions of $u \in [-M, M]$,

$$\mu_n(u) = b_n^{-1} n^{-1/2}\big(M_n(z_1 + h_n u) - M_n(z_1)\big). \tag{2.53}$$

Let $k_n^{(1)}$ be as in (2.29). The following two lemmas relate μ_n to $k_n^{(1)}$ and describe the strong limiting behavior of the sequence $\{\mu_n: n \ge 1\}$ as $n \to \infty$.

LEMMA 2.4. *Assume that F is continuous in a neighborhood of $z_1 \in (0, \Theta)$, that G is continuous at z_1 and that the derivative $f(x) = F'(x)$ of F at $x = z_1$ exists. Then, under* (H1) *and* (H2),

$$\sup_{u \in [-M, M]} |\mu_n(u) - k_n^{(1)}(u)| = O\big(h_n^{1/2}\big) \to 0 \quad \text{a.s.} \tag{2.54}$$

PROOF. In view of (2.29) and (2.52)–(2.53), we see that

$$\mu_n(u) = k_n^{(1)}(u) + b_n^{-1} n^{1/2} \int_z^{z + h_n u} (H_{n-}(t) - H_-(t))\, d\Lambda(t).$$

Hence, by an application of the Chung (1949) law of the iterated logarithm to the left-continuous empirical process $n^{1/2}(H_{n-}(t) - H_-(t))$, we obtain that, almost surely ultimately as $n \to \infty$,

$$\sup_{u \in [-M,M]} |\mu_n(u) - k_n^{(1)}(u)| \le b_n^{-1}(\log_2 n)^{1/2}\{\Lambda(z_1 + h_n M) - \Lambda(z_1 - h_n M)\}$$
$$= O(h_n^{1/2}) \to 0,$$

which is (2.54). □

LEMMA 2.5. *Under the assumptions of Lemma* 2.4, *the sequence* $\{\mu_n\}$ *is almost surely relatively compact in* $\mathbb{B}([-M, M])$ *with limit set equal to the set of all* $\mu \in \mathbb{B}([-M, M])$ *with*

$$\mu(u) = \int_0^u \Phi(s)\, ds \quad \text{for } -M \le u \le M, \text{ with } \int_{-M}^{M} \Phi^2(s)\, ds \le f(z_1)(1 - G(z_1)). \tag{2.55}$$

PROOF. The proof is straightforward by combining Remark 2.1 and Lemma 2.4. □

We will now make use of the following integral representation of the Kaplan–Meier empirical process [see, e.g., Gill (1980), page 37, Gu and Lai (1990), (1.23)–(1.24), page 164, (2.1), page 167, and (2.21), page 171]. We have, for all $x \le Z_{n,n} := \max\{Z_1, \dots, Z_n\}$,

$$\begin{aligned} n^{1/2}\left(\frac{F_n(x) - F(x)}{1 - F(x)}\right) &= n^{-1/2}\int_0^x \frac{dM_n(t)}{1 - H_-(t)} \\ &\quad + n^{-1/2}\int_0^x \left\{\left(\frac{1 - F_{n-}(t)}{1 - F(t)}\right)\frac{1}{1 - H_{n-}(t)} - \frac{1}{1 - H_-(t)}\right\} dM_n(t) \\ &=: \Pi_{n,1}(x) + \Pi_{n,2}(x) =: \Pi_n(x). \end{aligned} \tag{2.56}$$

Consider the following increment functions. Fix $z_1 \in (0, \Theta)$, $M > 0$, and set

$$\pi_n(u) = b_n^{-1}(\Pi_n(z_1 + h_n u) - \Pi_n(z_1)). \tag{2.57}$$

LEMMA 2.6. *Under the assumptions of Lemma* 2.4, *we have*

$$\sup_{u \in [-M,M]} \left|\pi_n(u) - \frac{\mu_n(u)}{1 - H(z_1)}\right| \to 0 \quad \text{a.s.} \tag{2.58}$$

PROOF. In our proof, we set $z = z_1$ for convenience. Since $Z_{n,n} \to \Theta$ a.s. as $n \to \infty$, for any $\theta < \Theta$ there exists almost surely an n_θ such that (2.56) holds for all $x \le \theta$ and $n \ge n_\theta$. We will therefore implicitly and without loss of generality assume that $n \ge n_\theta$ with $\theta > z + h_n M$. This allows us to set

$$\begin{aligned}\pi_{n,1}(u) &= b_n^{-1}(\Pi_{n,1}(z+h_n u) - \Pi_{n,1}(z))\\ &= b_n^{-1} n^{-1/2} \int_z^{z+h_n u} \frac{dM_n(t)}{1 - H_-(t)},\end{aligned} \tag{2.59}$$

and, by (2.50),

$$\begin{aligned}\pi_{n,2}(u) &= b_n^{-1}(\Pi_{n,2}(z+h_n u) - \Pi_{n,2}(z))\\ &= b_n^{-1} n^{-1/2} \int_z^{z+h_n u} \left\{ \left(\frac{1 - F_{n-}(t)}{1 - F(t)} \right) \frac{1}{1 - H_{n-}(t)} - \frac{1}{1 - H_-(t)} \right\} dM_n(t)\\ &= b_n^{-1} n^{1/2} \int_z^{z+h_n u} \left\{ \left(\frac{1 - F_{n-}(t)}{1 - F(t)} \right) \frac{1}{1 - H_{n-}(t)} - \frac{1}{1 - H_-(t)} \right\} dH_n^{(1)}(t)\\ &\quad - b_n^{-1} n^{1/2} \int_z^{z+h_n u} \left\{ \frac{1 - F_{n-}(t)}{1 - F(t)} - \frac{1 - H_{n-}(t)}{1 - H_-(t)} \right\} d\Lambda(t).\end{aligned} \tag{2.60}$$

It follows from (H1) and the assumption of local continuity of F that $F_-(t) = F(t)$ for all $t \in [z - h_n M,\ z + h_n M]$ and all n sufficiently large. This, in combination with the law of the iterated logarithm of Földes and Rejtő (1981) [see also Csörgő and Horváth (1983) and Gu and Lai (1990), (1.15)], implies that

$$\sup_{t \in [z - h_n M, z + h_n M]} \left| \frac{1 - F_{n-}(t)}{1 - F(t)} - 1 \right| = O(n^{-1/2}(\log_2 n)^{1/2}) \quad \text{a.s.} \tag{2.61}$$

Likewise, the Chung (1949) law of the iterated logarithm, when applied to the left-continuous empirical process $n^{1/2}(H_{n-}(t) - H_-(t))$, entails that

$$\sup_{t \in [z - h_n M, z + h_n M]} \left| \frac{1 - H_{n-}(t)}{1 - H_-(t)} - 1 \right| = O(n^{-1/2}(\log_2 n)^{1/2}) \quad \text{a.s.} \tag{2.62}$$

By combining (2.60), (2.61) and (2.62) with the nonnegativity of the measures $dH_n^{(1)}$ and $d\Lambda$, we obtain readily that, for some $A_n = O((\log_2 n)^{1/2})$ a.s., we have

$$\begin{aligned}\sup_{u \in [-M, M]} |\pi_{n,2}(u)| \le A_n\big\{&b_n^{-1}\big(H_n^{(1)}(z + h_n M) - H_n^{(1)}(z - h_n M)\big)\\ &+ b_n^{-1}\big(\Lambda(z + h_n M) - \Lambda(z - h_n M)\big)\big\}.\end{aligned} \tag{2.63}$$

Next, we observe from (2.29) taken with $z_1 = z$ and Theorem 2.1 that, almost surely as $n \to \infty$,

$$\begin{aligned}
&b_n^{-1}\big(H_n^{(1)}(z + h_n M) - H_n^{(1)}(z - h_n M)\big)\\
&\qquad = n^{-1/2}\{k_n^{(1)}(M) - k_n^{(1)}(-M)\}\\
&\qquad\quad + b_n^{-1}\{H^{(1)}(z + h_n M) - H^{(1)}(z - h_n M)\}\\
&\qquad = O(n^{-1/2}) + O\big(h_n^{1/2}(\log_2 n)^{-1/2}\big) = O\big(h_n^{1/2}(\log_2 n)^{-1/2}\big),
\end{aligned} \tag{2.64}$$

where we have made use of (H2). Recalling that $\Lambda(s) = -\log(1 - F(s))$ and $b_n = (2h_n \log_2 h)^{1/2}$, the assumption of existence of $f(x) = F'(x)$ at $x = z$ entails that

$$b_n^{-1}(\Lambda(z + h_n M) - \Lambda(z - h_n M)) = O\big(h_n^{1/2}(\log_2 n)^{-1/2}\big).$$

This, when combined with (2.63)–(2.64) and (H1), implies that

$$\sup_{u \in [-M,M]} |\pi_{n,2}(u)| = O\big(h_n^{1/2}\big) \to 0 \quad \text{a.s.} \tag{2.65}$$

To evaluate $\pi_{n,1}(u)$, we integrate by parts in (2.59), letting $dM_n(t) = d(M_n(t) - M_n(z))$, to obtain, via (2.53), that

$$\begin{aligned}
\pi_{n,1}(u) &= b_n^{-1} n^{-1/2} \left\{ \frac{M_n(z + h_n u) - M_n(z)}{1 - H_-(z + h_n u)} \right\}\\
&\quad - b_n^{-1} n^{-1/2} \int_0^u (M_n(z + h_n v) - M_n(z))\, d\left(\frac{1}{1 - H_-(z + h_n v)} \right)\\
&= \frac{\mu_n(u)}{1 - H_-(z + h_n u)} - \int_0^u \mu_n(v)\, d\left(\frac{1}{1 - H_-(z + h_n v)} \right).
\end{aligned} \tag{2.66}$$

In view of (2.66), Lemma 2.5, and the continuity of F, G and H at z, it is now immediate that

$$\sup_{u \in [-M,M]} \left| \pi_{n,1}(u) - \frac{\mu_n(u)}{1 - H(z)} \right| \to 0 \quad \text{a.s.} \tag{2.67}$$

The proof of (2.58) is completed by combining (2.65) and (2.67). □

We now make use of the notation in (1.7) and (1.8) by setting $\xi_n(u) = b_n^{-1}(a_n(z + h_n u) - a_n(z))$ with $z = z_1$ and $a_n(x) = n^{1/2}(F_n(x) - F(x))$, and let $k_n^{(1)}(u)$ be defined as in (2.29). The next lemma gives the final step in the proof of Theorem 1.2.

LEMMA 2.7. *Under the assumptions of Lemma* 2.4, *we have*

$$\sup_{u \in [-M,M]} \left| \xi_n(u) - \frac{k_n^{(1)}(u)}{1 - G(z_1)} \right| \to 0 \quad \text{a.s.} \tag{2.68}$$

PROOF. Let $z = z_1$. In view of (1.7)–(1.8) and (2.56)–(2.57), we have

$$
\begin{aligned}
\xi_n(u) &= b_n^{-1}\{(1 - F(z + h_n u))\Pi_n(z + h_n u) \\
&\qquad - (1 - F(z))\Pi_n(z)\} \qquad (2.69) \\
&= (1 - F(z))\Pi_n(u) + b_n^{-1}(F(z) - F(z + h_n u))\Pi_n(z + h_n u).
\end{aligned}
$$

By (2.56) and (2.61), we have, uniformly over $u \in [-M, M]$, $\Pi_n(z + h_n u) = O((\log_2 n)^{1/2})$, a.s., whereas (H1) and the existence of $f(x) = F'(x)$ at $x = z$ entail that $F(z) - F(z + h_n u) = O(h_n)$. Therefore, it follows from (2.69) that

$$
\sup_{u \in [-M,M]} |\xi_n(u) - (1 - F(z))\pi_n(u)| = O\big(h_n^{1/2}\big) \to 0 \quad \text{a.s.} \qquad (2.70)
$$

The conclusion (2.68) now follows directly from (2.70) when combined with (2.54), (2.58) and the fact, implied by (2.1), that $(1 - F(z))/(1 - H(z)) = 1/(1 - G(z))$. □

PROOF OF THEOREM 1.2. Assume first that the derivative $f(x) = F'(x)$ of F at $x = z \in (0, \Theta)$ exists and that G is continuous at z. If such is the case and under (H1) and (H2), we infer from Theorem 2.1 and Remark 2.1 that the sequence $\{k_n^{(1)}(u)/(1 - G(z))\colon n \geq 1\}$ of functions of $u \in [-M, M]$ is almost surely relatively compact in $\mathbb{B}([-M, M])$ with limit set consisting of all functions $h \in \mathbb{B}([-M, M])$ of the form

$$
\begin{aligned}
h(u) &= \int_0^u \frac{\phi(s)}{1 - G(z)}\, ds \\
&\qquad \text{for } -M \leq u \leq M, \text{ with } \int_{-M}^{M} \phi^2(s)\, ds \leq f(z)(1 - G(z)). \qquad (2.71)
\end{aligned}
$$

By setting $\Psi(s) = \phi(s)/(1 - G(z))$, we see that (2.71) is equivalent to (1.9).The remainder of the proof of the theorem under the assumption of existence of $f(x)$ at $x = z \in (0, \Theta)$ is completed by an application of Lemma 2.7 taken with $z = z_1$. In view of Remark 2.1, the proof of the theorem when only $f_-(x)$ and $f_+(x)$ exist at $x = z \in (0, \Theta)$ is very similar, so we omit the details. □

3. Applications. To illustrate how the results of the preceding section may be applied, we consider in the first place a continuous functional Γ, defined on a closed subset $\mathscr{S}$ of $\mathbb{B}([-M, M])$ for some $M > 0$ and satisfying the condition that $\xi_n \in \mathscr{S}$ for each $n \geq 1$. Introduce the statistic

$$
T_n = \Gamma(\xi_n), \qquad (3.1)
$$

with ξ_n defined as in (1.8). Letting θ denote a continuous function on $[-C, C]$, where $C \geq 0$ is a constant such that $0 \leq C \leq M$, simple examples of such functionals are given by

$$\Gamma_1(h) = |h(C)|, \qquad \Gamma_{2,\pm}(h) = \pm h(C), \qquad \Gamma_3(h) = \sup_{-C \leq t \leq C} |h(t)\theta(t)|, \tag{3.2}$$
$$\Gamma_{4,\pm}(h) = \sup_{-C \leq t \leq C} \pm h(t)\theta(t), \qquad \Gamma_5(h) = \int_{-C}^{C} h(t)\theta(t)\,dt.$$

The following theorem gives a description of the almost-sure limiting behavior of the sequence $\{T_n: n \geq 1\}$ under the above assumptions. Denote by $\mathbb{L}_M$ the set of all functions $h \in \mathbb{B}([-M, M])$ satisfying (1.9).

THEOREM 3.1. *Let $z \in (0, \Theta)$ and $M > 0$ be fixed. Assume that F is continuous in a neighborhood of z, that the derivative $f(x) = F'(x)$ of F at $x = z$ exists and that G is continuous at z. Then, under* (H1) *and* (H2), *the sequence $\{T_n: n \geq 1\}$ is almost surely relatively compact in $\mathbb{R}$ with limit set equal to the interval*

$$\left[\inf_{h \in \mathbb{L}_M} \Gamma(h), \sup_{h \in \mathbb{L}_M} \Gamma(h)\right]. \tag{3.3}$$

PROOF. The fact that $\{T_n: n \geq 1\}$ is almost surely relatively compact with limit set equal to $\Gamma(\mathbb{L}_M)$ is straightforward by Theorem 1.2. Since, by (1.9), the set $\mathbb{L}_M$ is compact and connected in $\mathbb{B}([-M, M])$, it follows that the image set $\Gamma(\mathbb{L}_M)$ of $\mathbb{L}_M$ by the continuous mapping Γ is a closed interval, so that (3.3) is immediate. □

We will now show that Theorem 1.1 is a consequence of Theorem 3.1.

PROOF OF THEOREM 1.1. Making use of (K1) and (K2), we first choose $M > 0$ in such a way that $K(u) = 0$ for all $|u| \geq M/2$, and then rewrite (1.2)–(1.3) as

$$\begin{aligned} f_n(z) - \mathbb{E}f_n(z) &= h_n^{-1}\int_{-M}^{M} K(u)\,d\{F_n(z + h_n u) - F_n(z) \\ &\qquad\qquad -F(z + h_n u) + F(z)\} \\ &= -h_n^{-1}\int_{-M}^{M} \{F_n(z + h_n u) - F_n(z) \\ &\qquad\qquad -F(z + h_n u) + F(z)\}\,dK(u) \\ &= -h_n^{-1} n^{-1/2} b_n \int_{-M}^{M} \xi_n(u)\,dK(u), \end{aligned} \tag{3.4}$$

where we have integrated by parts, which is rendered possible by (K1), and made use of the notations (1.7)–(1.8). Next, we define the functional Γ in (3.1) by setting, for each function $h \in \mathbb{B}([-M, M])$ of bounded variation on $[-M, M]$,

$$\Gamma(h) = -\int_{-M}^{M} h(u)\, dK(u), \tag{3.5}$$

Next, we apply Theorem 3.1 to obtain, via (3.4), that the limit set of the sequence

$$T_n = \left\{ \frac{nh_n}{2\log_2 n} \right\}^{1/2} (f_n(z) - \mathbb{E}f_n(z)), \qquad n = 1,2,\dots, \tag{3.6}$$

is as in (3.3). To conclude, we first integrate by parts to obtain, via (1.9), that

$$\begin{aligned} \sup_{h \in \mathbb{L}_M} \pm \Gamma(h) &= \sup_{h \in \mathbb{L}_M} \pm \int_{-M}^{M} K(u)\, dh(u) \\ &= \sup_{h \in \mathbb{L}_M} \left\{ \pm \int_{-M}^{M} K(u)\Psi(u)\, du \colon h(u) = \int_0^u \Psi(s)\, ds \right\}. \end{aligned} \tag{3.7}$$

To evaluate this last expression, it is convenient to set

$$c_- = \frac{1 - G_-(z)}{f_-(z)}, \qquad c_+ = \frac{1 - G(z)}{f_+(z)},$$

$$K^*(t) = \begin{cases} c_-^{-1/2} K(t), & \text{for } t < 0, \\ c_+^{-1/2} K(t), & \text{for } t \geq 0, \end{cases} \tag{3.8}$$

$$\Psi^*(t) = \begin{cases} c_-^{1/2} \Psi(t), & \text{for } t < 0, \\ c_+^{1/2} \Psi(t), & \text{for } t \geq 0. \end{cases}$$

By (3.8), $K(u)\Psi(u) = K^*(u)\Psi^*(u)$, and the inequality on the right-hand side of (1.9) is equivalent to $\int_{-M}^{M} \Psi^*(u)^2\, du \leq 1$. It follows that (3.7) may be rewritten as

$$\begin{aligned} \sup_{h \in \mathbb{L}_M} \pm \Gamma(h) &= \sup\left\{ \pm \int_{-M}^{M} K^*(u)\Psi^*(u)\, du \colon \int_{-M}^{M} (\Psi^*(u))^2\, du \leq 1 \right\} \\ &\leq \left\{ \int_{-M}^{M} (K^*(u))^2\, du \right\}^{1/2} \\ &= \left\{ \frac{f_-(z)}{1 - G_-(z)} \int_{-M}^{0} K^2(t)\, dt + \frac{f_+(z)}{1 - G(z)} \int_0^M K^2(t)\, dt \right\}^{1/2} \\ &=: L, \end{aligned} \tag{3.9}$$

where we have used (3.8) in combination with the Schwarz inequality. The particular choice of

$$\Psi^*(u) = \frac{\pm K^*(u)}{\left\{\int_{-M}^{M} \left(K^*(t)\right)^2 dt\right\}^{1/2}}$$

shows likewise that we have equality in (3.9). This, in combination with the just proven fact that the sequence T_n, as defined in (3.6), is almost surely relatively compact with limit set equal to the interval $[-L, L]$, with L as in (3.9), suffices for (1.4). □

REMARK 3.1. A simple modification of the arguments we have used in our proofs shows that, throughout, we may replace assumption (H1) by the following variant.

(H3) There exists a sequence $\{h_n^*, n \geq 1\}$ satisfying (H1), such that

$$0 < \liminf_{n\to\infty} \left(\frac{h_n}{h_n^*}\right) \leq \limsup_{n\to\infty} \left(\frac{h_n}{h_n^*}\right) < \infty.$$

This follows from the observation that, whenever the conclusion (1.9) of Theorem 1.2 holds for some sequence $\{h_n, n \geq 1\}$, then it also holds when, in the definition (1.8) of $\{\xi_n\}$, $\{h_n, n \geq 1\}$ is replaced by an arbitrary sequence $\{h_n^*, n \geq 1\}$ such that h_n/h_n^* is bounded away from zero and infinity.

REMARK 3.2. It is obvious from the arguments of the proof of Theorem 1.1 that the functional laws given in Theorems 1.2 and 3.1 may be used to describe the pointwise almost-sure limiting behavior of a large class of nonparametric estimators of local functionals of F. For example, if we choose $\Gamma = \Gamma_3$ and $\theta(t) = 1$ in (3.1) and (3.2), we obtain a description of the strong limiting behavior of the local modulus of continuity of the Kaplan–Meier process $\alpha_n = n^{1/2}(F_n(x) - F(x))$. Such applications, being readily obtained either as direct applications of our results or by arguments similar to those used to prove Theorem 1.1 given the conclusion of Theorem 3.1, are left to the reader.

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